10.1 Sequences and Summation Notation

Sequence

An infinite sequence a_n is a function whose domain is the set of positive integers.

A finite sequence is a function whose domain is the set of the first n positive integers. The values $a_1, a_2, a_3, \ldots$ are called the terms of the sequence.

The notation a_n (read as " a sub n") represents the nth term (or general term) of the sequence.

Write the first four terms of the sequence whose general term is given.

1) $a_n = \dfrac{(-1)^n}{3^n - 1}$ 1) ____________

$$a_1 = \frac{(-1)^1}{3^1 - 1} = \frac{-1}{3-1} = -\frac{1}{2} \quad ; \quad a_2 = \frac{(-1)^2}{3^2 - 1} = \frac{1}{9-1} = \frac{1}{8}$$

$$a_3 = \frac{(-1)^3}{3^3 - 1} = \frac{-1}{27-1} = -\frac{1}{26} \quad ; \quad a_4 = \frac{(-1)^4}{3^4 - 1} = \frac{1}{81-1} = \frac{1}{80}$$

Factorial Notation

If n is a positive integer, the notation n ! (read " *n* factorial ") is the product of all positive integers from *n* down through 1.

$n! = n(n-1)(n-2)(n-3)\ldots 1$; $0! = 1$(By definition)

Write the first four terms of the sequence whose general term is given.

2) $a_n = \dfrac{2^n}{(n-1)!}$ 2) ____________

$$a_1 = \frac{2^1}{(1-1)!} = \frac{2}{0!} = \frac{2}{1} = 2 \quad ; \quad a_2 = \frac{2^2}{(2-1)!} = \frac{4}{1!} = \frac{4}{1} = 4$$

$$a_3 = \frac{2^3}{(3-1)!} = \frac{8}{2!} = \frac{8}{2} = 4 \quad ; \quad a_4 = \frac{2^4}{(4-1)!} = \frac{16}{3!} = \frac{16}{6} = \frac{8}{3}$$

Summation Notation

The sum of the first *n* terms of a sequence is represented by the summation notation

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + a_4 + \ldots + a_n,$$

where *i* is the index of summation, *n* is the upper limit of summation, and 1 is the lower limit of summation.

Find the indicated sum.

3) $\sum_{i=1}^{4} [6 - (-1)^i] = [6 - (-1)^1] + [6 - (-1)^2] + [6 - (-1)^3] + [6 - (-1)^4]$ 3) ____________

$= 7 + 5 + 7 + 5 = 24$

Express the sum using summation notation.

4) $1^3 + 2^3 + 3^3 + \ldots + 7^3$ 4) ____________

The sum has seven terms, each of the form i^3, starting at $i = 1$ and ending at $i = 7$.

Thus, $1^3 + 2^3 + 3^3 + \ldots + 7^3 = \sum_{i=1}^{7} i^3$.

5) $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots + \frac{1}{243}$ 5) ____________

The sum has five terms, each of the form $\frac{1}{3^{i-1}}$, starting at $i = 1$ and ending at $i = 6$.

Thus, $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots + \frac{1}{243} = \sum_{i=1}^{6} \frac{1}{3^{i-1}}$.

10.1 Exercises pg 1010

(7, 19, 27, 35, 39, 49, 81) (11, 20, 28, 36, 40, 51, 84)

10.2 Arithmetic Sequence

Arithmetic Sequence

→ An *arithmetic sequence* is a sequence in which each term after the first differs from the preceding term by a constant amount.

→ The difference between consecutive terms is called the *common difference* of the sequence.

→ The common difference, *d*, is found by subtracting any term from the term that directly follows it. So, $d = a_n - a_{n-1}$.

Find the common difference and the fifth term of the arithmetic sequence.

6) 8, 3, –2, –7, . . . 6) ____________

The common difference is: $d = a_2 - a_1 = 3 - 8 = -5$.

$a_5 = a_4 + d = -7 + (-5) = -12$.

Write the first four terms of the arithmetic sequence.

7) $a_1 = \frac{3}{4}$; $d = -\frac{1}{4}$ 7) ____________

$a_2 = a_1 + d = \frac{3}{4} + \left(-\frac{1}{4}\right) = \frac{2}{4} = \frac{1}{2}$

$a_3 = a_2 + d = \frac{1}{2} + \left(-\frac{1}{4}\right) = \frac{1}{4}$ $a_4 = a_3 + d = \frac{1}{4} + \left(-\frac{1}{4}\right) = 0$

General Term of an Arithmetic Sequence

The nth term of an arithmetic sequence with first term a_1 and common difference d is $a_n = a_1 + (n - 1)d$.

Find the indicated term of the sequence with the given first term, a_1, and common difference, d.

8) Find a_{50} when $a_1 = -32$, $d = 4$. 8) ____________

$a_{50} = -32 + (50 - 1)(4) = -32 + (196) = 164$

Write a formula for the general term (the nth term) of the arithmetic sequence. Then use the formula for a_n to find a_{20}, the 20th term of the sequence.

9) 6, 1, –4, –9, . . . 9) ____________

$a_n = a_1 + (n - 1)d = 6 + (n - 1)(-5) = 6 - 5n + 5 = -5n + 11$.

$a_{20} = -5(20) + 11 = -100 + 11 = -89$.

The Sum of the First *n* Term of an Arithmetic Sequence

The sum, S_n, of the first *n* terms of an arithmetic sequence is given by

$$S_n = \frac{n}{2}(a_1 + a_n).$$

Find the indicated sum.

10) **Find the sum of the first 60 terms of the arithmetic sequence: −2, −11, −20, −29, . . .** 10) ____________

$a_{60} = -2 + (60 - 1)(-9) = -2 + (-531) = -533.$

$S_{60} = \frac{60}{2}[-2 + (-533)] = 30(-535) = -16{,}050.$

The number of even positive integers between any two positive odd integers *a* and *b* is $n = \frac{b - a}{2}$.

The number of odd positive integers between any two positive even integers *a* and *b* is $n = \frac{b - a}{2}$.

Find the indicated sum.

11) **Find the sum of the odd integers between 30 and 54.** 11) ____________

The number of odd positive integers between 30 and 54 is $n = \frac{54 - 30}{2} = 12.$ $S_{12} = \frac{12}{2}(31 + 53) = 6(84) = 504.$

Find the indicated sum.

12) $\sum_{i=1}^{49} (5i - 9)$ 12) ____________

$a_1 = 5(1) - 9 = 5 - 9 = -4$ $a_{49} = 5(49) - 9 = 245 - 9 = 236$

$S_{49} = \frac{49}{2}[-4 + 236] = \frac{49}{2}(232) = 5684.$

10.2 Exercises pg 1020

(3, 19, 25, 36, 41, 45, 59) (7, 21, 29, 37, 43, 47, 60)

10.3 Geometric Sequence

→ A *geometric sequence* is a sequence in which each term after the first is obtained by multiplying the preceding term by a constant amount.

→ The amount is called the *common ratio* of the sequence.

→ The common ratio, r , is found by dividing any term by the term that directly precedes it. So, $r = \frac{a_n}{a_{n-1}}$.

If the given sequence is geometric, find the common ratio and the fifth term of the sequence.

13) 6, –12, 24, –48, . . . $r = \frac{a_2}{a_1} = -2$ $a_5 = a_4 \cdot r = -48(-2) = 96.$ 13) ____________

Write the first four terms of the geometric sequence.

14) $a_1 = 3$; $r = -\frac{1}{4}$ $a_2 = a_1 \cdot r = 3\left(-\frac{1}{4}\right) = -\frac{3}{4}.$ 14) ____________

$a_3 = a_2 \cdot r = \left(-\frac{3}{4}\right)\left(-\frac{1}{4}\right) = \frac{3}{16}.$ $a_4 = a_3 \cdot r = \left(\frac{3}{16}\right)\left(-\frac{1}{4}\right) = -\frac{3}{64}.$

General Term of a Geometric Sequence

The nth term of a geometric sequence with first term a_1 and common ratio r is

$a_n = a_1 r^{n-1}$.

Use the formula for the general term (the nth term) of a geometric sequence to find the indicated term of the sequence with the given first term, a_1, and common ratio, r.

15) Find a_8 when $a_1 = -4$, $r = -2$. 15) ____________

$a_8 = (-4)(-2)^{8-1} = (-4)(-2)^7 = (-4)(-128) = 512.$

16) Find a_8 when $a_1 = 2000$, $r = -\frac{1}{2}$. 16) ____________

$a_8 = (2000)\left(-\frac{1}{2}\right)^{8-1} = (2000)\left(-\frac{1}{2}\right)^7 = (2000)\left(-\frac{1}{128}\right) = -\frac{125}{8}.$

Write a formula for the general term (the nth term) of the geometric sequence.Then use the formula for a_n to find a_{10}, the 10th term of the sequence.

17) $-3, -6, -12, -24, -48, \ldots$ 17) ___________

$a_n = a_1r^{n-1} = (-3)(2)^{n-1}$.

$a_{10} = (-3)(2)^{10-1} = (-3)(2)^9 = (-3)(512) = -1536$.

18) $4, 1, \frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \ldots$ $a_n = a_1r^{n-1} = (4)\left(\frac{1}{4}\right)^{n-1}$. 18) ___________

$a_{15} = (4)\left(\frac{1}{4}\right)^{10-1} = (4)\left(\frac{1}{4}\right)^9 = 4\left(\frac{1}{262{,}144}\right) = \frac{1}{65{,}536}$.

The Sum of the First *n* Term of a Geometric Sequence

The sum, S_n, of the first *n* terms of a geometric sequence is given by

$$S_n = \frac{a_1(1 - r^n)}{1 - r},$$ **with first term a_1 and common ratio r.**

Find the indicated sum. Use the formula for the sum of the first n terms of a geometric sequence.

19) **Find the sum of the first 10 terms of the geometric sequence:** 19) ___________
$2, -8, 32, -128, \ldots$

$$S_{10} = \frac{2\left(1 - (-4)^{10}\right)}{1 - (-4)} = \frac{2(1 - 1048576)}{1 + 4} = \frac{2(-1048575)}{5} = -419{,}430.$$

Find the indicated sum. Use the formula for the sum of the first n terms of a geometric sequence.

20) $\sum_{i=1}^{10} 6(2)^i$ 20) ___________

$$S_{10} = \frac{12\left(1 - 2^{10}\right)}{1 - 2} = \frac{12(1 - 1024)}{-1} = \frac{12(-1023)}{-1} = 12{,}276.$$

10.3 Practice Exercises pg 1035
(1, 11, 19, 27, 34) (3, 13, 22, 29, 35)

10.5 The Binomial Coefficient

Definition of a Binomial Coefficient

For nonnegative integers n and r, with n ≥ r, the expression $\binom{n}{r}$, (read " *n* above *r* ") is called a binomial coefficient and is defined by $\binom{n}{r} = \dfrac{n!}{r!\,(n-r)!}$.

Evaluate the given binomial coefficient.

21) $\binom{10}{5} = \dfrac{10!}{5!\,(12-5)!} = \dfrac{10!}{5!\,7!} = \dfrac{3628800}{120 \cdot 5040} = 6$ 21) ___________

Expanding Binomials: The Binomial Theorem

For any positive integer *n*,

$$(a+b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 + \ldots + \binom{n}{n} a^0 b^n$$

Use the Binomial Theorem to expand the binomial and express the result in simplified form.

22) $(x+2)^4$; $a = x \quad b = 2$ 22) ___________

$= \binom{4}{0} x^4 + \binom{4}{1} x^3 \cdot 2^1 + \binom{4}{2} x^2 \cdot 2^2 + \binom{4}{3} x^1 \cdot 2^3 + \binom{4}{4} 2^4$

$= x^4 + 8x^3 + 24x^2 + 32x + 16$

23) $(2x - y)^4$; $a = 2x \quad b = -y$ 23) ___________

$= \binom{4}{0} (2x)^4 + \binom{4}{1} (2x)^3 \cdot (-y)^1 + \binom{4}{2} (2x)^2 \cdot (-y)^2$

$+ \binom{4}{3} (2x)^1 \cdot (-y)^3 + \binom{4}{4} (-y)^4$

$= 16x^4 - 32x^3y + 24x^2y^2 - 8xy^3 + y^4$

Finding a Particular Term in a Binomial Expansion

The rth term of th expansion $(a + b)^n$ is $\binom{n}{r-1} a^{[n-(r-1)]} b^{r-1}$

Find the term indicated in the expansion.

24) $(3x - 2y)^7$; 4th term 24) ____________

The 4th term is: $\binom{7}{3}(3x)^4 \cdot (-2y)^3$

$= 35(81x^4)(-8y^3) = -22680x^4y^3$

25) $(3x - 2)^9$; 5th term 25) ____________

The 5th term is: $\binom{9}{4}(3x)^5 \cdot (-2)^4$

$= 126(243x^5)(16) = 489{,}888x^5$

26) $(x^3 - y^5)^{11}$; 6th term 26) ____________

The 6th term is: $\binom{11}{5}(x^3)^6 \cdot (-y^5)^5 = -462x^{18}y^{25}$

10.5 Exercises pg 1001 (15, 33, 39, 43) (21, 37, 41, 45)